X-INNER AUTOMORPHISMS OF CROSSED PRODUCTS AND SEMI-INVARIANTS OF HOPF ALGEBRAS[†]

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ABSTRACT

Let $R * G$ be a crossed product of the group G over the prime ring R and assume that $R * G$ is also prime. In this paper we study units q in the Martindale ring of quotients $Q_0(R * G)$ which normalize both R and the group of trivial units of $R * G$. We obtain quite detailed information on their structure. We then study the group of X-inner automorphisms of $R * G$ induced by such elements. We show in fact that this group is fairly close to the group of automorphisms of $R * G$ induced by certain trivial units in $Q_0(R) * G$. As an application we specialize to the case where $R = U(L)$ is the enveloping algebra of a Lie algebra L. Here we study the semi-invariants for L and G which are contained in $Q_0(R * G)$ and we obtain results which extend known properties of $U(L)$. Finally, every cocommutative Hopf algebra H over an algebraically closed field of characteristic 0 is of the form $H = U(L) * G$. Thus we also obtain information on the semi-invariants for H contained in $Q_0(H)$.

§1. Introduction

In the first part of this paper, we consider general crossed products $R * G$ with R prime and we describe certain X-inner automorphisms of these rings. Our results extend known facts about X-inner automorphisms of group rings [11, 12]. Recall that an automorphism of a prime ring T is X-inner if it becomes inner when extended to the Martindale quotient ring $Q_0(T)$. Other basic definitions and notation can be found in [10].

The goal here is to describe the X-inner automorphisms of $R * G$ which normalize both R and the group $\mathfrak{G}$ of trivial units of $R * G$. Observe that if q is a unit of $Q_0(R * G)$ which gives rise to such an X-inner automorphism, then q

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induces a group automorphism σ on $\mathfrak{G}/\mathfrak{U} \simeq G$. Here $\mathfrak{U}$ is the group of units of R. To be precise we have $q^{-1}R\bar{x}q = R\overline{x^\sigma}$ for all $x \in G$.

For convenience we will use a symmetric version of the Martindale ring of quotients. If R is prime, we set

$$Q_s(R) = \{f \in Q_0(R) \mid fI \subseteq R \text{ for some } 0 \neq I \lhd R\}.$$

It is easy to see that $S = Q_s(R)$ is a subring of $Q_0(R)$ containing R. Furthermore if the crossed product $R * G$ is given, then there exists a natural extension to a crossed product $S * G$.

Since we are interested in prime rings, we use [10, Theorem 2.8] which says that $R * G$ will be prime if $G_{\text{inn}} \cap \Delta^+ = 1$. Here $\Delta(G)$ is the f.c. center of G, that is, the set of elements of G having only finitely many conjugates, $\Delta^+(G)$ is its torsion subgroup and G_{inn} is the normal subgroup of G consisting of those automorphisms which are X-inner on R. In view of [12, Lemma 4.1.6] this condition is equivalent to $G_{\text{inn}} \cap \Delta$ being torsion free abelian. We remark that this condition is sufficient for primeness but not necessary, as can be seen in Example 4.9.

The main result here is

THEOREM A. *Let $R * G$ be a crossed product with R prime and $G_{\text{inn}} \cap \Delta^+(G) = 1$. Let q be a unit of $Q_0(R * G)$ and σ an automorphism of G with $q^{-1}R\bar{x}q = R\overline{x^\sigma}$ for all $x \in G$. Let $S * G$ be the natural extension of $R * G$ with $S = Q_s(R)$. Then*

(i) *$\sigma = \sigma_1\sigma_2$ where σ_1 centralizes a subgroup of G of finite index and σ_2 is an inner automorphism of G;*

(ii) *$q = \alpha^{-1}\beta s\bar{g}$, where*

$$0 \neq \alpha \in \mathbf{Z}(S * G) \subseteq S * (G_{\text{inn}} \cap \Delta(G)), \qquad \beta \in S * (G_{\text{inn}} \cap \Delta(G))$$

centralizes S, s is a unit of S normalizing R and $g \in G$.

Furthermore, we take a closer look at the element β and the interrelations between the various terms in (ii) above. With this we obtain the following description of the group of X-inner automorphisms of $R * G$ normalizing both R and $\mathfrak{G}$.

THEOREM B. *Let $R * G$ be a crossed product with R prime and $G_{\text{inn}} \cap \Delta^+(G) = 1$. Let $S * G$ be the natural extension of $R * G$ with $S = Q_s(R)$. Let $\mathfrak{X}$ be the group of X-inner automorphisms of $R * G$ normalizing both R and the group $\mathfrak{G}$ of trivial units of $R * G$. If $\mathfrak{X}_0$ is the subgroup of $\mathfrak{X}$ consisting of those automorphisms induced by trivial units of $S * G$, then $\mathfrak{X}/\mathfrak{X}_0$ is a torsion abelian group.*

We remark that the group of trivial units of $S * G$ need not normalize $R * G$. Thus $\mathfrak{X}_0$ corresponds to a subgroup of this group.

In the second part of this paper we specialize to the case in which L is a Lie algebra over a field k, $R = U(L)$ is its universal enveloping algebra and RG is a prime skew group ring. Let $0 \neq q \in Q_0(RG)$. We say that q is a semi-invariant for L and G if there exists a linear functional $\mu : L \to k$ with

$$[l, q] = \mu(l) q \qquad \text{for all } l \in L$$

and a linear character $\lambda : G \to k$ with

$$x^{-1} q x = \lambda(x) q \qquad \text{for all } x \in G.$$

In other words, q is a common eigenvector for the natural actions of L and of G on $Q_0(RG)$. We define the semicenter SZ of RG to be the linear span of all semi-invariants for L and G contained in RG itself. SZ is a subalgebra of RG.

If q is a semi-invariant, then q is easily seen to be a unit of $Q_0(RG)$ normalizing both R and the group $\mathfrak{G}$ of trivial units of RG. Thus the previous results apply and are a first step towards proving

THEOREM C. *Let RG be a k-algebra skew group ring with $R = U(L)$, the universal enveloping algebra of a finite dimensional Lie algebra over k. Assume that RG is prime and k has characteristic zero. Then*

(i) *SZ is a commutative integral domain;*

(ii) *every semi-invariant for L and G in $Q_0(RG)$ is a quotient of semi-invariants contained in SZ.*

This result also holds if the skew group ring RG is replaced by a crossed product $R * G$ provided we make some mild assumption on the twisting. A more precise formulation will be given in Section 4. In addition we will obtain a good deal of information about the individual semi-invariants valid in all characteristics.

Theorem C extends known results about enveloping algebras and group algebras. In the case of enveloping algebras of characteristic 0 (that is, G is trivial), it is known that every nonzero ideal of $U(L)$ contains a nonzero semi-invariant. This is a result of Moeglin [8] if k is algebraically closed, and is due independently to Malliavin [6] and Ginsburg [4] for arbitrary k of characteristic 0. The fact that every semi-invariant q for L in $Q_{cl}(U(L))$ is a quotient of semi-invariants in $U(L)$ is a well-known and trivial consequence of this result. Indeed the set $I = \{r \in U(L) \mid rq \in U(L)\}$ is a nonzero ideal of $U(L)$, so it contains a semi-invariant b. Then $bq = a \in U(L)$ is also a semi-invariant and thus $q = b^{-1}a$ is a quotient of semi-invariants.

It is not known whether this fact is true for $U(L)$ of characteristic $p > 0$. The characteristic 0 proofs in [4], [6], and [8] are all quite difficult.

For the case of group algebras $k[G]$ (that is, L is trivial), it follows from [11, Lemma 3] that if q is a semi-invariant for G in $Q_0(k[G])$, then $q = c\alpha$ where α is a semi-invariant for G in $k[G]$ and $c \in C$, the extended centroid of $k[G]$. By Formanek's theorem [3], $c = b^{-1}a$ for a and b in the center of $k[G]$ and we obtain $q = b^{-1}(a\alpha)$, a quotient of semi-invariants in $k[G]$.

Finally we remark that the definition of semi-invariant as given above is really quite natural. If L is a Lie algebra over k, then the k-algebra skew group ring $U(L)G$ is a cocommutative Hopf algebra. Indeed if k is algebraically closed of characteristic 0, then Kostant's theorem (see [13, Theorems 1 and 2] and [14, Theorems 8.1.5 and 13.0.1]) asserts that every such Hopf algebra H is of this form. Now given a Hopf algebra H and an H-module algebra A, one can define, as in [1], the idea of an inner action of H on A. With $H = U(L)G$ and $A = H$ or $Q_0(H)$, this action is precisely the one considered above. Thus the semi-invariants for L and G are precisely the common eigenvectors for the inner action of H on itself or on $Q_0(H)$. See [1] for complete details.

In the course of our work, we will need to study automorphisms of G centralizing a subgroup of finite index. The following is a slight extension of [12, Lemma 1].

Lemma 1.1. *Let σ be an automorphism of the arbitrary group G with $|G : \mathbf{C}_G(\sigma)| = n < \infty$. Then σ acts trivially on G/Δ and Δ/Δ^+.*

(i) *If (G,σ) is finite, then σ has finite order.*

(ii) *If (G,σ) is torsion free, then σ^n is conjugation by some $h \in (G,\sigma)$.*

(iii) *σ^m is conjugation by h for some integer $m > 0$ and some $h \in (G,\sigma)$.*

Proof. Let $W = \mathbf{C}_G(\sigma)$, let T be a right transversal for W in G and set $B = \{t^{-1}t^\sigma \mid t \in T\}$. Note that for any $x, y \in G$ we have $x^{-1}x^\sigma = y^{-1}y^\sigma$ if and only if $x \in Wy$. From this we conclude that B is independent of the choice of transversal and that $|B| = |T| = |G : W| = n$. Now if $w \in W$, then $w^{-1}Tw$ is also a transversal for W and since $w^\sigma = w$ we see that $B^w = B$. Hence since $|B| < \infty$ and $|G : W| < \infty$ we conclude that $B \subseteq \Delta(G)$. Set $H = (G,\sigma)$ so that, by definition, $H = \langle B \rangle$. Then H is a finitely generated subgroup of Δ and it is a standard group theoretic fact that $H \lhd G$. Clearly σ acts trivially on G/H and hence on G/Δ. Furthermore σ acts on the torsion free abelian group Δ/Δ^+, centralizing a subgroup of finite index. Thus σ also centralizes Δ/Δ^+.

(i) If H is finite, say of order m, then σ normalizes each coset Hg and since $|Hg| = m$ we have $\sigma^{m!} = 1$.

(ii) If H is torsion free, then since $H \subseteq \Delta(G)$ we know that H is abelian. Since σ acts on H and centralizes a subgroup of finite index, it follows that σ centralizes H. Define $\mu : G \to H$ by $x^\sigma = x\mu(x)$ for $x \in G$. Since σ centralizes H it follows that $x^{\sigma^i} = x\mu(x)^i$ for all integers i.

Since σ is an automorphism of G we have

$$xy \cdot \mu(xy) = (xy)^\sigma = x^\sigma y^\sigma = x\mu(x) \cdot y\mu(y)$$

and we obtain the cocycle equation $\mu(xy) = \mu(x)^y \mu(y)$. Set

$$h = \prod_{t \in T} \mu(t) = \prod_{b \in B} b \in H.$$

Then h is independent of the choice of transversal since H is abelian. If we fix $y \in G$ and multiply the cocycle equation over all $x \in T$ we conclude, since Ty is also a transversal, that $h = h^y \mu(y)^n$. Thus for all $y \in G$ we have

$$y^{\sigma^n} = y\mu(y)^n = y(h^y)^{-1}h = h^{-1}yh$$

and σ^n is the inner automorphism induced by $h \in H = (G, \sigma)$.

(iii) Let H_1 be a characteristic torsion free subgroup of H of finite index. Then $H_1 \lhd G$ and σ acts on $\bar{G} = G/H_1$ with $(\bar{G}, \sigma) = H/H_1$ finite. Thus σ has finite order in its action on $\bar{G}$, by (i), or equivalently, for some integer $m > 0$, we have $(G, \sigma^m) \subseteq H_1$. We can now apply (ii) to the automorphism σ^m to deduce the result.

§2. Theorem A

The goal of this section is to prove Theorem A. In the course of that proof we will have to deal with the Martindale ring of quotients of $R * N$ for various normal subgroups N of G. For this we require a large ring in which they all embed and, as we see below, the maximal ring of quotients of $R * G$ will suffice.

Let R be any ring. If D is a right ideal of R and $a \in R$, then we set $a^{-1}D = \{r \in R \mid ar \in D\}$. By definition, D is dense if and only if $l_R(a^{-1}D) = 0$ for all $a \in R$. Recall that the maximal right ring of quotients $Q_m(R)$ is the set of all equivalence classes $[D, f]$ of module homomorphisms $f : D_R \to R_R$ from the dense right ideals of R to R. The following slightly extends a result of [3].

Lemma 2.1. *If $R * G$ is given, then there is a natural inclusion $Q_m(R) * G \subseteq Q_m(R * G)$. In particular if $H \lhd G$ then $Q_m(R * H) \subseteq Q_m(R * G)$.*

PROOF. Let D be a dense right ideal of R. Then clearly $l_{R*G}(D)=0$ and $\bar{x}^{-1}D\bar{x}$ is also dense for any $x \in G$. Furthermore if $\alpha = \Sigma_1^n r_i\bar{x}_i \in R * G$ then we have

$$\alpha^{-1}D(R*G) \supseteq \bigcap_1^n \bar{x}_i^{-1}(r_i^{-1}D)\bar{x}_i.$$

It now follows that $D(R*G)$ is a dense right ideal of $R*G$.

Next if $f: D_R \to R_R$, then the natural extension $\hat{f}: D(R*G) \to R*G$ given by $\hat{f}(\Sigma d_i\bar{x}_i) = \Sigma f(d_i)\bar{x}_i$ is easily seen to be an $R*G$-module homomorphism. We therefore obtain an embedding of $Q_m(R)$ into $Q_m(R*G)$ via the map $[D,f] \to [D(R*G),\hat{f}]$.

Recall that the embedding of $R*G$ into $Q_m(R*G)$ is given by left multiplication. From this it is easy to see that if $x \in G$ then $\bar{x}$ normalizes $Q_m(R)$. Furthermore the sum $\Sigma_{x\in G} Q_m(R)\bar{x}$ is direct since if $\Sigma \bar{x}f_x = 0$ then by evaluating this function on a common dense domain for the f_x's we conclude that each $f_x = 0$. It is now clear that

$$Q_m(R*G) \supseteq \bigoplus \sum_{x\in G} Q_m(R)\bar{x} = Q_m(R)*G \supseteq R*G.$$

Finally if $H \lhd G$, then $R*G = (R*H)*(G/H)$ so $Q_m(R*G) \supseteq Q_m(R*H)$.

As indicated earlier, we will use a symmetric version of the Martindale ring of quotients. Let R be a prime ring and let $Q_0(R)$ be its Martindale ring of quotients as given in [10, §2]. Then we set

$$Q_s(R) = \{f \in Q_0(R) \mid fI \subseteq R \text{ for some } 0 \neq I \lhd R\}.$$

Thus $Q_s(R)$ is a subring of $Q_0(R)$ containing R and we have

LEMMA 2.2. *Let R be a prime ring.*

(i) *If q is a unit of $Q_0(R)$ with $q^{-1}Rq = R$, then $q \in Q_s(R)$.*

(ii) *If $f \in Q_s(R)$, $0 \neq I \lhd R$ and $fI = 0$, then $f = 0$.*

(iii) $Q_s(R) \subseteq Q_m(R)$.

PROOF. (i) If $Iq \subseteq R$ then $q(q^{-1}Iq) \subseteq R$.

(ii) Let $0 \neq J \lhd R$ with $Jf \subseteq R$ and observe that $0 = J(fI) = (Jf)I$. Since R is prime and $I \neq 0$ we deduce that $Jf = 0$ and then that $f = 0$.

(iii) This is clear from (ii) above and the definition of $Q_s(R)$ since every nonzero two-sided ideal of a prime ring is dense.

LEMMA 2.3. *Let H be an ordered subgroup of G and let $R*G$ be a crossed product with R prime. If α and β are nonzero elements with $\alpha \in Q_s(R)*H$ and $\beta \in Q_m(R*G)$ then $\alpha R\beta \neq 0$. In particular, if α centralizes R then $\alpha\beta \neq 0$.*

PROOF. There exists a nonzero ideal I of R with $0 \neq I\alpha \subseteq R * H$ and there exists a dense right ideal D of $R * G$ with $0 \neq \beta D \subseteq R * G$. Thus it suffices to assume that $\alpha \in R * H$ and $\beta \in R * G$. Furthermore since $R * G$ is a free left $R * H$-module, we may assume in addition that $\beta \in R * H$. Finally let x be the maximal element in the support of α and let y be the maximal element in the support of β. If $\alpha = \bar{x}a + \cdots$ and $\beta = b\bar{y} + \cdots$, then $aRb \neq 0$ implies that $\alpha R\beta \neq 0$.

We remark that any torsion free abelian group is ordered so the above applies to such subgroups H.

The proof of Theorem A requires Δ-methods. Let G be a group and K a subset of G. For convenience we say that K is a c.c.-subset (coset complement subset) if

$$K = G \setminus \bigcup_{1}^{n} H_i x_i$$

with each subgroup H_i of infinite index in G. The key fact we need about these sets is

LEMMA 2.4. *Let K, K' be c.c.-subsets of G and let W be a subgroup of G of finite index. Then $K \cap K' \cap W \neq \emptyset$.*

PROOF. It is clear that $K \cap K'$ is also a c.c.-subset of G and then that $(K \cap K') \cap W$ is a c.c.-subset of W. Thus we need only observe from [13, Lemma 4.2.1] that c.c.-subsets are nonempty.

If H is a subgroup of G, then its almost centralizer $\mathbf{D}_G(H)$ is defined by

$$\mathbf{D}_G(H) = \{x \in G \mid |H : \mathbf{C}_H(x)| < \infty\}.$$

It is clear that $\mathbf{D}_G(H)$ is a subgroup of G normalized by H. If $|G : H| < \infty$, then $\mathbf{D}_G(H) = \mathbf{D}_G(G) = \Delta(G)$, the f.c. center of G.

Let $R * G$ be a crossed product and let H be a subgroup of G. Then we have the natural projection map $\pi_H : R * G \to R * H$ given by

$$\pi_H\left(\sum_{x \in G} r_x \bar{x}\right) = \sum_{x \in H} r_x \bar{x}.$$

This is easily seen to be an $R * H$-bimodule homomorphism. If $H = 1$ we write $\operatorname{tr} = \pi_1$ and if $H = \Delta(G)$ then we use $\theta = \pi_\Delta$. Finally if V is any subset of G, we let $R * V$ denote the set of all elements $\alpha \in R * G$ with support, $\operatorname{Supp} \alpha$, contained in V.

LEMMA 2.5. *Let V be a finite subset of G and let H be a subgroup of G with*

$D = \mathbf{D}_G(H)$. *Then there is a c.c.-subset K of H with the following property. Suppose*

$$\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n \in R * V \subseteq R * G$$

and let $x \in K$. If

$$\alpha_1 \bar{x} \beta_1 + \alpha_2 \bar{x} \beta_2 + \cdots + \alpha_n \bar{x} \beta_n = 0$$

then

$$\pi_D(\alpha_1)\bar{x}\beta_1 + \pi_D(\alpha_2)\bar{x}\beta_2 + \cdots + \pi_D(\alpha_n)\bar{x}\beta_n = 0.$$

PROOF. We use the same proof as that of [10, Lemma 1.3]. Let $x \in H$ and suppose that the first equation is satisfied but that the second is not. Then there exist $v_1 \in V \backslash D$, $v_2 \in V \cap D$ and $v_3, v_4 \in V$ with $v_1 x v_3 = v_2 x v_4$. Thus $x^{-1} v_1 x = (x^{-1} v_2 x) v_4 v_3^{-1}$ and since $v_2 \in D$ and V is finite there are only finitely many possibilities for the right hand term. This shows that x must belong to a finite union of cosets of $\mathbf{C}_H(v_1)$ for the various $v_1 \in V \backslash D$. By definition of D, each such $\mathbf{C}_H(v_1)$ has infinite index in H.

We now begin the proof of Theorem A. For this we require some results from reference [10] but in a slightly different form than given there. In that paper, the crossed product $R * G$ is studied by extending it to $Q_0(R) * G$, while here we work with the smaller ring $Q_s(R) * G$. However in view of Lemma 2.2(i), it is easy to see that the results of [10] are equally valid in the present context.

We are given $q \in Q_0(R * G)$ and $\sigma \in \operatorname{Aut} G$ with $q^{-1} R \bar{x} q = R \overline{x^\sigma}$. We work in $Q_m(R * G)$ which contains the relevant subrings we require. Note that $q \in Q_s(R * G) \subseteq Q_m(R * G)$. Furthermore if $S = Q_s(R)$, then $Q_m(R * G) \supseteq S * G \supseteq R * G$. We also use the notation and conclusion of [10, Lemma 2.3]. Thus we let $E = \mathbf{C}_{S*G}(S)$ and we have $S * G_{\text{inn}} = S \otimes_C E$ with $E = C^t[G_{\text{inn}}]$, some twisted group algebra over the field $C = \mathbf{Z}(S)$. Observe that S is a prime ring and that the projection maps π_H extend naturally to $S * G$.

LEMMA 2.6. *With the above notation we have*

(i) *there exists $0 \neq \alpha \in E$ with* $\operatorname{tr} \alpha = 1$ *and $\alpha q \in S * G$;*

(ii) *if $Y = \{y \in G \mid \bar{y} q^{-1}$ induces an X-inner automorphism on $R\}$ then Y is a coset of G_{inn}, $x^{-1} Y x^\sigma = Y$ for all $x \in G$ and σ normalizes G_{inn};*

(iii) *if $\alpha \in E$ with $\alpha q \in S * G$ and if $y \in Y$, then $\alpha q = \beta s \bar{y} \in S * Y$ with $\beta \in E$ and s a unit of S which normalizes R.*

PROOF. By definition of $Q_0(R * G)$ there exists $0 \neq I \triangleleft R * G$ with $Iq \subseteq R * G$. By [10, Lemma 2.4] I contains a nonzero element $\alpha' = a\alpha$ with $a \in R$

and $\alpha \in E$. Furthermore, a close look at the proof of that lemma shows that $\operatorname{tr}\alpha = 1$. Set $\beta' = \alpha' q \in R * G$ and write $\beta' = \sum b_x \bar{x}$.

If $r \in R$ then since $\alpha \in E$ we have

$$\beta'(ra)^q = a\alpha q(ra)^q = a\alpha(ra)q$$

$$= a(ra)\alpha q = ar\beta'.$$

Thus for each $b_x \neq 0$ we have

$$b_x \bar{x}(ra)^q = arb_x \bar{x}$$

so

$$b_x(ra)^{q\bar{x}^{-1}} = arb_x$$

for all $r \in R$. By [10, Lemma 2.2] we conclude that there exists a unit $s_x \in S$ such that $b_x = as_x$ and conjugation by s_x induces the automorphism ${}^{q\bar{x}^{-1}}$ on R. The latter implies that conjugation by $s_x\bar{x}$ induces the automorphism ^{q} on R.

Set $\beta = \sum s_x \bar{x}$ so that $\beta' = a\beta$ and $r\beta = \beta r^q$ for all $r \in R$. Since $r(\alpha q) = (\alpha q)r^q$ we therefore have

$$0 = (\alpha' q - \beta')r^q = ar(\alpha q - \beta)$$

for all $r \in R$. Lemma 2.3 with $H = 1$ yields $\alpha q - \beta = 0$ so $\alpha q \in S * G$.

This completes the proof of (i) and we also note an additional observation. Since q is a unit and $\alpha \neq 0$ we have $\beta \neq 0$. If $g \in \operatorname{Supp}\beta \neq \emptyset$, then s_g is a unit of S and conjugation by $s_g\bar{g}$ induces the automorphism ^{q} on R. This shows that the set

$$Y = \{y \in G \mid \bar{y}q^{-1} \text{ induces an } X\text{-inner automorphism of } R\}$$

is nonempty and then it is clearly a coset of G_{inn} in G.

Now let $x \in G$ and notice that $\bar{y}q^{-1}$ is X-inner on R if and only if $\bar{x}^{-1}\bar{y}q^{-1}\bar{x}$ is X-inner. Observe that the latter element is, up to a unit in R, equal to $\bar{z}q^{-1}$ with $z = x^{-1}yx^\sigma$. Thus $y \in Y$ if and only if $z = x^{-1}yx^\sigma \in Y$ and we have $x^{-1}Yx^\sigma = Y$. Furthermore since Y is a coset of $G_{\text{inn}} \lhd G$, the latter implies that $x \in G_{\text{inn}}$ if and only if $x^\sigma \in G_{\text{inn}}$. Thus σ normalizes G_{inn} and (ii) follows.

Finally let α be any element of E with $\alpha q = \gamma \in S * G$ and let $y \in Y$. Then by definition of Y there exists a unit s of S normalizing R such that $q(s\bar{y})^{-1}$ centralizes R. Thus

$$\beta = \gamma(s\bar{y})^{-1} = \alpha q(s\bar{y})^{-1} \in E \subseteq S * G_{\text{inn}}$$

and $\gamma = \beta(s\bar{y}) \in (S * G_{\text{inn}})\bar{y} = S * Y$.

The next result contains part (i) of Theorem A and from this point on we assume the full hypothesis of that theorem. In particular $G_{\text{inn}} \cap \Delta^+ = 1$ so $G_{\text{inn}} \cap \Delta$ is torsion free abelian. Set $G_0 = \mathbf{C}_G(\sigma)$.

LEMMA 2.7. *With the above notation we have $\sigma = \sigma_1\sigma_2$ where σ_1 centralizes a subgroup of G of finite index and σ_2 is an inner automorphism of G. Now suppose in addition that $|G : G_0| < \infty$. Then*

(i) $Y \cap \Delta(G) \neq \varnothing$;

(ii) *if $\alpha, \gamma \in S * G$ with $\alpha q = \beta \in S * G$ and $q\gamma \in S * G$, then there exists a c.c.-subset K of G with*

$$\theta(\beta)\bar{x}^q\gamma = \theta(\alpha)\bar{x}q\gamma$$

for all $x \in K$.

PROOF. It is convenient to form a slightly larger ring. Let $\langle t \rangle$ be an infinite cyclic group and form the skew group ring $T = (R * G)\langle t \rangle$ where t acts on $R * G$ via conjugation by q. It is clear that $\{\bar{x}t^i \mid \text{all } x \in G, \text{all integers } i\}$ is an R-basis for T. Hence since $q^{-1}R\bar{x}q = R\overline{x^{\sigma}}$, it follows that T is in fact a crossed product over R of the group $\tilde{G} = G \times_{\sigma} \langle t \rangle$. Here t acts on G via the automorphism σ. We will work in $Q_m(T)$, a ring large enough to contain all relevant elements. Note that $T = R * \tilde{G} \subseteq S * \tilde{G}$.

Let $\alpha, \gamma \in S * G$ with $\alpha q = \beta \in S * G$ and $q\gamma = \delta \in S * G$. Notice that qt^{-1} centralizes $R * G$ so that $\bar{x}q = qt^{-1}\bar{x}t$ for all $x \in G$. This yields

$$\alpha\bar{x}\delta = \alpha\bar{x}q\gamma = \alpha q t^{-1}\bar{x}t\gamma = \beta t^{-1}\bar{x}t\gamma.$$

To repeat, for all $x \in G$ we have

$$\alpha\bar{x}\delta = (\beta t^{-1})\bar{x}(t\gamma),$$

an equation in $S * \tilde{G}$. If $D = \mathbf{D}_{\tilde{G}}(G)$, then by Lemmas 2.4 and 2.5 there exists a c.c.-subset $K \neq \varnothing$ of G such that for all $x \in K$

$$\pi_D(\alpha)\bar{x}\delta = \pi_D(\beta t^{-1})\bar{x}(t\gamma).$$

Observe that $D \cap G = \mathbf{D}_G(G) = \Delta(G)$ and that $\alpha \in S * G$ so $\pi_D(\alpha) = \theta(\alpha)$. Also $\delta = q\gamma$ so we have

$$(*) \qquad \theta(\alpha)\bar{x}(q\gamma) = \pi_D(\beta t^{-1})\bar{x}(t\gamma)$$

for all $x \in K$.

We can now quickly prove (ii). Suppose $|G : G_0| < \infty$. Then clearly $t \in D$ so

$$\pi_D(\beta t^{-1}) = \pi_D(\beta)t^{-1} = \theta(\beta)t^{-1}$$

since $\beta \in S * G$. Thus (*) yields

$$\theta(\alpha)\bar{x}(q\gamma) = \theta(\beta)(t^{-1}\bar{x}t)\gamma = \theta(\beta)\bar{x}^{q}\gamma$$

for all $x \in K$.

We return now to the general situation. By Lemma 2.6(i) there exists $0 \neq \alpha \in E$ with $\alpha q = \beta \in S * G$. Furthermore $\operatorname{tr}\alpha = 1$ so $\theta(\alpha) \neq 0$. Since $q \in Q_s(R * G)$ there exists $0 \neq \gamma \in R * G$ with $q\gamma \in R * G$. Thus equation (*) holds for some $x \in K \neq \emptyset$. Observe that $\alpha \in E \subseteq S * G_{\text{inn}}$ so $0 \neq \theta(\alpha) \in S * (G_{\text{inn}} \cap \Delta)$ and $G_{\text{inn}} \cap \Delta(G)$ is torsion free abelian by the hypothesis of Theorem A. Also $\theta(\alpha)$ clearly commutes with S and $\bar{x}q\gamma \neq 0$. We therefore conclude from Lemma 2.3 that $\theta(\alpha)\bar{x}(q\gamma) \neq 0$ and hence by (*) that $\pi_D(\beta t^{-1}) \neq 0$.

The latter implies that there exists $g \in \operatorname{Supp}\beta \subseteq G$ with $gt^{-1} \in D$. Then $tg^{-1} \in D$ and $t = tg^{-1} \cdot g$. Let σ_1 denote the action of tg^{-1} on G and let σ_2 denote the inner automorphism induced by g. Since t acts via σ on G, the above yields $\sigma = \sigma_1\sigma_2$. Furthermore since $tg^{-1} \in D$, we see that σ_1 centralizes a subgroup of G of finite index.

It remains to prove (i) and again we assume that $|G : G_0| < \infty$. Again this implies that $t \in D$ so the above yields $0 \neq \pi_D(\beta t^{-1}) = \theta(\beta)t^{-1}$ and hence $\theta(\beta) \neq 0$. But $\beta \in S * Y$ by Lemma 2.6(iii) so $\theta(\beta) \in S * (Y \cap \Delta)$ and we conclude that $Y \cap \Delta(G) \neq \emptyset$.

The goal now is to sharpen Lemma 2.6 in case $|G : G_0| < \infty$.

LEMMA 2.8. *Let $|G : G_0| < \infty$. Then there exists an element $0 \neq \alpha \in E$ such that $\alpha q \in S * G$ and $0 \neq \theta(\alpha)$ is a central element of $S * G$.*

PROOF. By Lemma 2.6(i) there exists $\alpha \in E$ with $\theta(\alpha) \neq 0$ and $\alpha q \in S * G$. Among all such elements we choose α so that $|\operatorname{Supp}\theta(\alpha)|$ is of minimal nonzero size. If $\alpha = \sum a_x\bar{x}$, then each $a_x\bar{x}$ is also in E and if $a_x \neq 0$ then a_x is a unit of S. Since $\theta(\alpha) \neq 0$ there exists $z \in \operatorname{Supp}\theta(\alpha)$. But then $\alpha' = (a_z\bar{z})^{-1}\alpha$ satisfies $\operatorname{tr}\alpha' = 1$, $\alpha' q \in S * G$, $\alpha' \in E$ and $|\operatorname{Supp}\theta(\alpha')| = |\operatorname{Supp}\theta(\alpha)|$. Thus we may assume now that $\operatorname{tr}\alpha = 1$. Furthermore, as indicated above, we have $\theta(\alpha) \in E \cap (S * \Delta) \subseteq S * (G_{\text{inn}} \cap \Delta)$.

Let G_1 denote the centralizer in G of $\operatorname{Supp}\theta(\alpha)$ so that $|G : G_1| < \infty$ and let $g \in G_0 \cap G_1$. Then $\alpha^{\bar{g}} \in E$, $\operatorname{Supp}\theta(\alpha^{\bar{g}}) = \operatorname{Supp}\theta(\alpha)$ since $g \in G_1$ and $\operatorname{tr}\alpha^{\bar{g}} = 1$. Furthermore since $g \in G_0$ we have $q^{\bar{g}} = qu$ for some unit u of R so $\alpha^{\bar{g}}q = (\alpha q)^{\bar{g}}u^{-1} \in S * G$. Thus $\gamma = \alpha - \alpha^{\bar{g}}$ satisfies $\gamma \in E$, $\gamma q \in S * G$ and $|\operatorname{Supp}\theta(\gamma)| < |\operatorname{Supp}\theta(\alpha)|$ since $\operatorname{tr}\gamma = 0$. By the minimality of $|\operatorname{Supp}\theta(\alpha)|$ we conclude that $0 = \theta(\gamma) = \theta(\alpha) - \theta(\alpha)^{\bar{g}}$. Thus $\theta(\alpha) \in E$ commutes with $S * (G_0 \cap G_1)$.

By Lemma 2.6(ii), σ normalizes $G_{\rm inn}$. Hence σ acts on the torsion free abelian group $G_{\rm inn}\cap\Delta$, centralizing a subgroup of finite index. It follows that σ must centralize $G_{\rm inn}\cap\Delta$ so $G_0\supseteq G_{\rm inn}\cap\Delta$. Furthermore $G_1\supseteq G_{\rm inn}\cap\Delta$ since the latter group is abelian. Thus we see that $\theta(\alpha)\in \mathbf{Z}(S*(G_{\rm inn}\cap\Delta))$ and hence, since $|G:G_0\cap G_1|<\infty$, the same is true for the finitely many $\bar{G}$-conjugates of $\theta(\alpha)$. In particular all these conjugates commute and if δ denotes the product of the distinct conjugates different from $\theta(\alpha)$, then $\delta\theta(\alpha)\in\mathbf{Z}(S*G)$. Again since $G_{\rm inn}\cap\Delta$ is torsion free abelian it follows from Lemma 2.3 that the center of $S*(G_{\rm inn}\cap\Delta)$ is a domain. Thus $\delta\theta(\alpha)\neq 0$.

Finally set $\tilde{\alpha}=\delta\alpha$ so that $\theta(\tilde{\alpha})=\delta\theta(\alpha)$ is a nonzero central element of $S*G$. Since $\delta\in\mathbf{Z}(S*(G_{\rm inn}\cap\Delta))\subseteq E$ we have $\tilde{\alpha}\in E$ and $\tilde{\alpha}q=\delta(\alpha q)\in S*G$. The result follows.

The next result essentially proves Theorem A.

LEMMA 2.9. *Let $|G:G_0|<\infty$. Then there exists a nonzero central element α of $S*G$ with $\alpha q\in S*(Y\cap\Delta)\subseteq S*G$.*

PROOF. Let α be as in Lemma 2.8 so that $\theta(\alpha)$ is a nonzero central element of $S*G$, $\alpha\in E$ and $\alpha q=\beta\in S*G$. For each $x\in G$ we define $\tau_x=\bar{x}^{-1}\theta(\beta)\bar{x}^q$. By Lemma 2.6(iii), $\beta\in S*Y$ so $\theta(\beta)\in S*(Y\cap\Delta)$. By Lemma 2.6(ii), $x^{-1}Yx^\sigma=Y$ and by Lemma 1.1, $x^{-1}\Delta x^\sigma=\Delta$ since $x^\sigma\in x\Delta$. Thus we see that $x^{-1}(Y\cap\Delta)x^\sigma=Y\cap\Delta$ and $\tau_x\in S*(Y\cap\Delta)$. By Lemma 2.7(i), $Y\cap\Delta\neq\varnothing$ and we choose a fixed element h in this set. Thus clearly $Y\cap\Delta=G_{\rm inn}h\cap\Delta=(G_{\rm inn}\cap\Delta)h$ and we can write each τ_x as $\tau_x=\tau'_x\bar{h}$ with $\tau'_x\in S*(G_{\rm inn}\cap\Delta)$.

Let $s\in S$. Then since $\alpha\in E$ we have

$$s\beta=s\alpha q=\alpha qs^q=\beta s^q$$

and thus $s\theta(\beta)=\theta(\beta)s^q$. Hence

$$\begin{aligned}s\tau_x&=\bar{x}^{-1}(\bar{x}s\bar{x}^{-1})\theta(\beta)\bar{x}^q\\&=\bar{x}^{-1}\theta(\beta)(\bar{x}s\bar{x}^{-1})^q\bar{x}^q\\&=\bar{x}^{-1}\theta(\beta)\bar{x}^q s^q=\tau_x s^q\end{aligned}$$

and therefore

$$s\tau'_x=\tau'_x s^{q\bar{h}^{-1}}$$

for all $s\in S$.

By definition of $Q_s(R*G)$ there exists $0\neq J\lhd R*G$ with $qJ\subseteq R*G$. Let $0\neq\gamma\in J$. Then Lemma 2.7(ii) implies that there is a c.c.-subset $K(\gamma)$ of G with

$$\theta(\beta)\bar{x}^q\gamma=\theta(\alpha)\bar{x}q\gamma$$

for all $x \in K(\gamma)$. Since $\theta(\alpha)$ is central, multiplying on the left by $\bar{x}^{-1}$ yields

$$\tau_x\gamma = \theta(\alpha)q\gamma$$

for all $x \in K(\gamma)$. The goal here is to show that the various τ_x's are equal.

Suppose first that $x, y \in K(\gamma)$. Then $\tau_x\gamma = \theta(\alpha)q\gamma = \tau_y\gamma$ yields $0 = (\tau'_x - \tau'_y)\bar{h}\gamma$. Furthermore by multiplying on the left by $s \in S$ we obtain

$$0 = s(\tau'_x - \tau'_y)\bar{h}\gamma = (\tau'_x - \tau'_y)s^{q\bar{h}^{-1}}\bar{h}\gamma$$

so $(\tau'_x - \tau'_y)S\bar{h}\gamma = 0$. Since $\bar{h}\gamma \neq 0$ and $\tau'_x - \tau'_y \in S * (G_{\mathrm{inn}} \cap \Delta)$ with the latter group torsion free abelian, we deduce from Lemma 2.3 that $\tau'_x - \tau'_y = 0$. Therefore $\tau_x = \tau'_x\bar{h} = \tau'_y\bar{h} = \tau_y$.

We have shown that τ_x is a constant for all $x \in K(\gamma)$. Now let γ' be a second element of $J\backslash 0$. Then τ_x is also a constant for all $x \in K(\gamma')$. But by Lemma 2.4, $K(\gamma) \cap K(\gamma') \neq \varnothing$. We therefore conclude that τ_x is a constant for all $\gamma \in J\backslash 0$ and all $x \in K(\gamma)$. If $\tau \in S * G$ denotes this constant element, we then have $(\tau - \theta(\alpha)q)\gamma = 0$ for all $\gamma \in J$. Since $R * G$ is prime, $0 \neq J \triangleleft R * G$ is a dense right ideal. Hence since $\tau - \theta(\alpha)q \in Q_m(R * G)$ we see that $\tau - \theta(\alpha)q = 0$. Since $0 \neq \theta(\alpha)$ is central in $S * G$ and $\tau \in S * (Y \cap \Delta)$, the lemma is proved.

It is now a simple matter to prove the following slight extension of Theorem A.

THEOREM 2.10. *Let $R * G$ be a crossed product with R prime and $G_{\mathrm{inn}} \cap \Delta^+(G) = 1$. Let q be a unit of $Q_0(R * G)$ and σ an automorphism of G with $q^{-1}R\bar{x}q = R\overline{x^\sigma}$ for all $x \in G$. Let $S * G$ be the natural extension of $R * G$ with $S = Q_s(R)$. Then*

(i) *$\sigma = \sigma_1\sigma_2$ where σ_1 centralizes a subgroup of G of finite index and σ_2 is an inner automorphism of G;*

(ii) *$q = \alpha^{-1}\beta s\bar{g}$ where*

$$0 \neq \alpha \in \mathbf{Z}(S * G) \subseteq S * (G_{\mathrm{inn}} \cap \Delta(G)), \qquad \beta \in S * (G_{\mathrm{inn}} \cap \Delta(G))$$

centralizes S, s is a unit of S normalizing R and $g \in G$;

(iii) *if $G_0 = \mathbf{C}_G(\sigma)$, then G_0 normalizes the coset $(G_{\mathrm{inn}} \cap \Delta(G))g$;*

(iv) *if $|G : G_0| < \infty$, then g above is contained in $\Delta(G)$.*

PROOF. Part (i) follows from Lemma 2.7. For (ii), suppose first that $|G : G_0| < \infty$. Then by Lemmas 2.9 and 2.6(iii), there exists $0 \neq \alpha \in \mathbf{Z}(S * G)$ with $\alpha q = \beta s\bar{y}$ with $\beta \in E \cap (S * \Delta) \subseteq S * (G_{\mathrm{inn}} \cap \Delta)$, $y \in G$ and s a unit of S normalizing R. Observe that $S * G$ is prime so α is regular in $S * G$. Hence since $\alpha \in$

$Q_m(R*G)$ it follows that α is central and invertible in $Q_m(R*G)$. Thus we have $q=\alpha^{-1}\beta s\bar{y}$. On the other hand, if σ is arbitrary, then (i) implies that $\sigma=\sigma_1\sigma_2$ where σ_2 is the inner automorphism induced by say $h\in G$. But then the above applies to $\tilde{q}=q\bar{h}^{-1}$ so $q\bar{h}^{-1}=\tilde{q}=\alpha^{-1}\beta s\bar{y}$. Thus $q=\alpha^{-1}\beta s\bar{y}\bar{h}$ also has the appropriate form and (ii) is proved.

Now let $x\in G_0=\mathbf{C}_G(\sigma)$. Then $\bar{x}^{-1}q\bar{x}=qu$ for some unit $u\in R$ so we have $\beta^{\bar{x}}(s\bar{g})^{\bar{x}}=\beta(s\bar{g})u$ and thus

$$(\operatorname{Supp}\beta)^x g^x=(\operatorname{Supp}\beta)g.$$

In particular since $\operatorname{Supp}\beta\subseteq G_{\mathrm{inn}}\cap\Delta(G)$ we see that G_0 normalizes the coset $(G_{\mathrm{inn}}\cap\Delta(G))g$. Furthermore since $\operatorname{Supp}\beta\subseteq\Delta(G)$, we see that $\{g^x\mid x\in G_0\}$ is finite. Thus if $|G:G_0|<\infty$, then $g\in\Delta(G)$.

We have therefore proved Theorem A. We remark that $S*G\subseteq Q_0(R)*G$ so that the elements α,β,s of (ii) above are all contained in the larger crossed product $Q_0(R)*G$. Furthermore, they are easily seen to enjoy the analogous properties in that ring. In particular, we have $\alpha\in\mathbf{Z}(Q_0(R)*G)$, β centralizes $Q_0(R)$ and $q=\alpha^{-1}\beta s\bar{g}$.

§3. Theorem B

In this section we amplify Theorem A. For the most part, we are concerned with the nature of the β term in the formula $q=\alpha^{-1}\beta s\bar{g}$. This is of course also related to the automorphism σ_1 of G which occurs in the formula $\sigma=\sigma_1\sigma_2$ and which centralizes a subgroup of G of finite index. With a proper understanding of β, it is a simple matter to prove Theorem B. We continue with the notation and assumptions of the proof of Theorem A. In particular $R*G$ is a crossed product with R prime and $G_{\mathrm{inn}}\cap\Delta^+(G)=1$. Thus $R*G$ is prime. Again we set $S=Q_s(R)$.

LEMMA 3.1. *In the notation of Theorem* 2.10 *we have*

(i) *q acts on R as $s\bar{g}$ does;*

(ii) *α is a unit of $Q_0(R*G)$ and in fact an element of the extended centroid of $R*G$;*

(iii) *βs is a unit of $Q_0(R*G)$ and there exists an automorphism τ of G with*

$$(\beta s)^{-1}R\bar{x}(\beta s)=R\overline{x^{\tau}}$$

for all $x\in G$;

(iv) *β is a unit of $Q_0(S * G)$ and*

$$\beta^{-1} S\bar{x}\beta = S\overline{x^{\tau}}$$

for all $x \in G$;

(v) *we can assume that* $\operatorname{tr}\beta = 1$.

PROOF. (i) This is clear since both α and β centralize $S \supseteq R$.

(ii) Since $\alpha \in S * G$ there exists $0 \neq I \lhd R$ with $I\alpha \subseteq R * G$. But then, since α centralizes $R * G$ we conclude first that $(R * G)I(R * G)\alpha \subseteq R * G$ so $\alpha \in Q_0(R * G)$ and then that α is contained in the extended centroid of $R * G$, namely the center of this quotient ring. Since the extended centroid is a field, α is a unit in $Q_0(R * G)$.

(iii) This is clear from (ii) since q, α and $\bar{g}$ are all units of $Q_0(R * G)$ normalizing both R and the group $\mathfrak{G}$ of trivial units of $R * G$.

(iv) Since s is a unit of S and β centralizes S, (iii) implies that $(S\bar{x})\beta = \beta(S\overline{x^{\tau}})$ for all $x \in G$. In particular $0 \neq \beta \in S * G$ is a normal element of the prime ring $S * G$. Thus β is a unit of $Q_0(S * G)$.

(v) Let $\beta = \Sigma s_x \bar{x}$ and let $h \in \operatorname{Supp}\beta$. Since $\beta \in E$, the centralizer of S, it follows that $s_h\bar{h}$ is invertible and also contained in E. Set $\beta' = \beta(s_h\bar{h})^{-1}$. Then $q = \alpha^{-1}\beta'(s_h\bar{h})(s\bar{g})$ has the appropriate form, $\beta' \in S * (G_{\text{inn}} \cap \Delta)$ since $h \in G_{\text{inn}} \cap \Delta$ and $\beta' \in E$. Since $\operatorname{tr}\beta' = 1$, the lemma is proved.

As we see below, the β terms with $\operatorname{tr}\beta = 1$ are particularly well behaved.

LEMMA 3.2. *Let β, s, τ be as above and set $W = \mathbf{C}_G(\tau)$. Then $|G : W| < \infty$, $W \supseteq G_{\text{inn}} \cap \Delta(G)$ and τ centralizes $G/(G_{\text{inn}} \cap \Delta)$. Furthermore if* $\operatorname{tr}\beta = 1$ *then $\beta \in \mathbf{Z}(S * W)$ and in particular $\beta \in \mathbf{Z}(S * (G_{\text{inn}} \cap \Delta))$.*

PROOF. It follows from Lemma 3.1(iii) that for all $x \in G$ there exists a unit u_x of S with $\bar{x}\beta = u_x\beta\overline{x^{\tau}}$. Thus we have

$$x(\operatorname{Supp}\beta) = (\operatorname{Supp}\beta)x^{\tau}.$$

Since $\operatorname{Supp}\beta$ is a finite subset of $G_{\text{inn}} \cap \Delta(G) \lhd G$, the above implies that $\{x^{\tau}x^{-1} \mid x \in G\}$ is a finite subset of G contained in $G_{\text{inn}} \cap \Delta$. Note that $x^{\tau}x^{-1} = y^{\tau}y^{-1}$ if and only if $y^{-1}x \in \mathbf{C}_G(\tau) = W$. We therefore conclude that $|G : W| < \infty$. Furthermore since these commutators $x^{\tau}x^{-1}$ are contained in $G_{\text{inn}} \cap \Delta$, it follows that τ centralizes $G/(G_{\text{inn}} \cap \Delta)$ and acts on $G_{\text{inn}} \cap \Delta$. By assumption, the latter group is torsion free abelian so we conclude from $|G : W| < \infty$ that τ centralizes $G_{\text{inn}} \cap \Delta$.

Now let $\operatorname{tr}\beta = 1$ and let $x \in W$ so that $x^{\tau} = x$. Then we have $\bar{x}\beta = u_x\beta\overline{x^{\tau}} = u_x\beta\bar{x}$. Since $\operatorname{tr}\beta = 1$, we see by comparing coefficients of $\bar{x}$ that $1 = u_x$ and hence that β centralizes $\bar{x}$. But we already know that β centralizes S so we conclude that β centralizes $S * W$. Since $\beta \in S * (G_{\text{inn}} \cap \Delta) \subseteq S * W$, the result follows.

We can now characterize the automorphisms τ of G which come from a βs term. Recall that $\mathfrak{G}$ is the group of trivial units of $R * G$. Thus $\mathfrak{G}$ contains $\mathfrak{U}$, the group of units of R, and $\mathfrak{G}/\mathfrak{U} \simeq G$. Furthermore we let $\tilde{\mathfrak{U}}$ be the group of units of $S = Q_s(R)$ which normalize R and we set $\tilde{\mathfrak{G}} = \tilde{\mathfrak{U}}\mathfrak{G}$. Thus $\tilde{\mathfrak{G}}$ is a subgroup of the group of trivial units of $S * G$ and again $\tilde{\mathfrak{G}}/\tilde{\mathfrak{U}} \simeq G$.

Proposition 3.3. *Let $R * G$ be a crossed product with R a prime ring and $G_{\text{inn}} \cap \Delta^+(G) = 1$. Let τ be an automorphism of G. Then τ is induced from a βs term with* $\operatorname{tr} \beta = 1$ *if and only if τ lifts to an automorphism $\tilde{\tau}$ of $\tilde{\mathfrak{G}}$ (induced by β) with*

(i) $|\tilde{\mathfrak{G}} : \mathbf{C}_{\tilde{\mathfrak{G}}}(\tilde{\tau})| < \infty$;

(ii) $\mathbf{C}_{\tilde{\mathfrak{G}}}(\tilde{\tau}) \supseteq \tilde{\mathfrak{U}}$ *and* $\mathbf{C}_{\tilde{\mathfrak{G}}}(\tilde{\tau})/\tilde{\mathfrak{U}} = \mathbf{C}_G(\tau)$;

(iii) $\tilde{\tau}$ *centralizes* $\tilde{\mathfrak{G}}/\mathbf{C}_{\tilde{\mathfrak{G}}}(S)$;

(iv) $\mathfrak{G}^{\tilde{\tau}} = \mathfrak{G}^u$ *for some* $u \in \tilde{\mathfrak{U}}$.

Proof. Assume first that βs exists. By Lemma 3.1(iii)(iv) since $s \in \tilde{\mathfrak{U}}$, β gives rise to an automorphism $\tilde{\tau}$ of $\tilde{\mathfrak{G}}$ centralizing $\tilde{\mathfrak{U}}$. Let $\mathfrak{W} = \mathbf{C}_{\tilde{\mathfrak{G}}}(\tilde{\tau})$ and $W = \mathbf{C}_G(\tau)$. Then $\mathfrak{W} \supseteq \tilde{\mathfrak{U}}$ and $\mathfrak{W}/\tilde{\mathfrak{U}} \subseteq \mathbf{C}_G(\tau) = W$ since fixed points map to fixed points. On the other hand, by Lemma 3.2, β centralizes $S * W$ and this clearly yields $\mathfrak{W}/\tilde{\mathfrak{U}} = W$. Since $|G : W| < \infty$, by Lemma 3.2, we therefore have $|\tilde{\mathfrak{G}} : \mathfrak{W}| < \infty$ and hence (i) and (ii) are satisfied. For (iii) observe that β centralizes S. Hence for any $g \in \tilde{\mathfrak{G}}$, $\beta^{-1} g \beta$ and g will act the same on S and thus $g^{\tilde{\tau}} g^{-1} = \beta^{-1} g \beta g^{-1} \in \mathbf{C}_{\tilde{\mathfrak{G}}}(S)$. Finally, βs normalizes $\mathfrak{G}$ so $\mathfrak{G}^{\tilde{\tau}} = \mathfrak{G}^{s^{-1}}$ and (iv) is proved.

For the converse, assume $\tilde{\tau}$ exists satisfying (i), (ii), (iii) and (iv). Let $\mathfrak{T}$ be a right transversal for $\mathfrak{W}$ in $\tilde{\mathfrak{G}}$ and set

$$\beta = \sum_{t \in \mathfrak{T}} t^{-1} t^{\tilde{\tau}} \in S * G.$$

Notice that β is independent of the choice of transversal. Indeed if $t \in \mathfrak{T}$ is replaced by wt with $w \in \mathfrak{W}$, then

$$(wt)^{-1}(wt)^{\tilde{\tau}} = t^{-1}(w^{-1} w^{\tilde{\tau}}) t^{\tilde{\tau}} = t^{-1} t^{\tilde{\tau}},$$

since $w^{\tilde{\tau}} = w$. Next for any $g \in \tilde{\mathfrak{G}}$

$$g^{-1} \beta g^{\tilde{\tau}} = \sum_{t \in \mathfrak{T}} g^{-1} t^{-1} t^{\tilde{\tau}} g^{\tilde{\tau}} = \sum_{t \in \mathfrak{T}} (tg)^{-1} (tg)^{\tilde{\tau}}$$

$$= \sum_{t \in \mathfrak{T} g} t^{-1} t^{\tilde{\tau}} = \beta$$

since $\bar{\mathfrak{T}}g$ is also a right transversal for $\mathfrak{W}$. In particular, this implies that W normalizes Supp β so since $|G:W|<\infty$ we have Supp $\beta \subseteq \Delta(G)$.

We now show that $\beta \neq 0$. Let $T=\{x_1,\ldots,x_n\}$ be a transversal for W in G. Then by (ii), $\mathfrak{T}=\{\bar{x}_1,\ldots,\bar{x}_n\}$ is a transversal for $\mathfrak{W}$. Note that for $x,y\in G$ we have $x^{-1}x^{\tau}=y^{-1}y^{\tau}$ if and only if $yx^{-1}\in W$. Thus we see that each $\bar{x}_i^{-1}\bar{x}_i^{\bar{\tau}}$ is contained in a distinct component $S\bar{g}$ with $g\in G$. Hence $\beta\neq 0$ and in fact by taking $x_1=1$ we see that $\operatorname{tr}\beta=1$.

By (iii) each $t^{-1}t^{\bar{\tau}}$ centralizes S so $\beta\in E\subseteq S*G_{\text{inn}}$. Finally by (iv) there exists $u\in\tilde{\mathfrak{U}}$ with $\mathfrak{G}^{\bar{\tau}}=\mathfrak{G}^{u}$. Thus if $s=u^{-1}$, then $0\neq\beta s\in S*(G_{\text{inn}}\cap\Delta)$, $\mathfrak{G}^{\bar{\tau}s}=\mathfrak{G}$ and, by the above,

$$(\beta s)g^{\bar{\tau}s}=g(\beta s)$$

for all $g\in\mathfrak{G}$. In particular since $s\in\tilde{\mathfrak{U}}$ and $\beta\in E$ we have the normalizing condition $(\beta s)(R*G)=(R*G)(\beta s)$. Now $\beta s\in S*G$ so there exists $0\neq I\lhd R$ with $I(\beta s)\subseteq R*G$. But then $(R*G)I(R*G)(\beta s)\subseteq R*G$ so we see that $\beta s\in Q_0(R*G)$. The normalizing condition then implies that βs is a unit of $Q_0(R*G)$ and the above formula yields

$$(\beta s)^{-1}R\bar{x}(\beta s)=R\bar{x}^{\bar{\tau}}=R\overline{x^{\tau}}$$

for all $x\in G$. This completes the proof.

We close this section with the

PROOF OF THEOREM B. Let $\mathfrak{X}$ denote the group of X-inner automorphisms of $R*G$ normalizing both R and $\mathfrak{G}$. Let $\mathfrak{X}_0$ be its subgroup consisting of those automorphisms induced by trivial units of $S*G$. We remark that some care is necessary since not every trivial unit of $S*G$ normalizes $R*G$.

By definition, any automorphism in $\mathfrak{X}$ is induced by a unit q of $Q_0(R*G)$ with $q^{-1}R\bar{x}q=R\overline{x^{\sigma}}$ for some $\sigma\in\operatorname{Aut}G$. In particular Theorem A applies and we have $q=\alpha^{-1}\beta s\bar{g}$ with $\alpha\in\mathbf{Z}(S*G)$. But then the action of α is trivial so, by Lemma 3.1(ii), it suffices to delete the α term and assume that $q=\beta s\bar{g}$. Furthermore by Lemma 3.1(v) we may assume that $\operatorname{tr}\beta=1$.

Observe that, by Lemma 3.1(iv) and Proposition 3.3, β is a unit of $Q_0(S*G)$ normalizing $\tilde{\mathfrak{G}}=\tilde{\mathfrak{U}}\mathfrak{G}$. Thus q also normalizes $\tilde{\mathfrak{G}}$ and $q\equiv\beta \bmod \tilde{\mathfrak{G}}$.

We first show that $\mathfrak{X}/\mathfrak{X}_0$ is abelian. To this end, let $q_1=\beta_1s_1\bar{g}_1$ and $q_2=\beta_2s_2\bar{g}_2$ be given. Then $\beta_1,\beta_2\in\mathbf{Z}(S*(G_{\text{inn}}\cap\Delta))$, by Lemma 3.2, so they commute. Since $q_i\equiv\beta_i \bmod \tilde{\mathfrak{G}}$ and $\tilde{\mathfrak{G}}$ is normalized by these elements, it follows that q_1 and q_2 commute modulo $\tilde{\mathfrak{G}}$. Thus the commutator (q_1,q_2) is a trivial unit of $S*G$ which induces an automorphism on $R*G$ and hence it corresponds to an element of $\mathfrak{X}_0$. We conclude that $\mathfrak{X}_0\supseteq\mathfrak{X}'$ and hence also that $\mathfrak{X}_0\lhd\mathfrak{X}$.

Finally we show that $\mathfrak{X}/\mathfrak{X}_0$ is torsion. To this end, let $q = \beta s\bar{g}$ be given. By Proposition 3.3, β induces an automorphism $\bar{\tau}$ on $\mathfrak{G}$ centralizing a subgroup of finite index. Thus by Lemma 1.1 applied to $\bar{\tau}$ and $\mathfrak{G}$, for some integer $n > 0$ and some $t \in (\mathfrak{G}, \bar{\tau})$, $\bar{\tau}^n$ is conjugation by t. Furthermore, since $\bar{\tau}$ centralizes $\mathfrak{G}/\mathbf{C}_{\mathfrak{G}}(S)$, we see that $t \in \mathbf{C}_{\mathfrak{G}}(S)$. Thus since β also centralizes S, it follows that β^n and t agree in their action on both $\mathfrak{G}$ and S so they agree in their action on $S * G$. Now $q \equiv \beta \bmod \mathfrak{G}$ and $\mathfrak{G}$ is normalized by both these elements so $q^n \equiv \beta^n \bmod \mathfrak{G}$. It therefore follows that q^n acts on $R * G$ like an element of $\mathfrak{G}$, so q^n corresponds to an automorphism in $\mathfrak{X}_0$ and the theorem is proved.

This is of course an extension of [12, Theorem 3(iii)] which considered ordinary group algebras.

§4. Theorem C

In this section, we specialize to the case in which L is a Lie algebra over a field k and $R = U(L)$ is its universal enveloping algebra. Then R is a domain and we study k-algebra crossed products $R * G$. The goal is to prove an extension of Theorem C.

Since the units of $U(L)$ are precisely the nonzero elements of k, and hence central in $R * G$, any such crossed product determines a homomorphism $G \to \mathrm{Aut}(R)$. Automorphisms of particular interest are the translations, namely the maps $\tau : R \to R$ determined by $\tau(l) = l + \mu(l)$ where $l \in L$ and $\mu(l) \in k$. Clearly $\mu : L \to k$ must be a linear functional. Let T denote the set of all $g \in G$ which act like translations. Then T is a subgroup of G and $T \supseteq G_{\mathrm{inn}}$ by [9, Theorem 1], a fact we will use freely throughout this section. Note that T need not be normal in G.

Proposition 4.1. *Let $R = U(L)$ be the universal enveloping algebra of the Lie algebra L over k and let $R * G$ be a k-algebra crossed product. Let T be the translation subgroup of G.*

(i) *If $T \cap \Delta^+(G) = 1$, then $R * G$ is prime.*

(ii) *If $R * G$ is a prime skew group ring and* char $k = 0$, *then $T \cap \Delta^+(G) = 1$.*

Proof. (i) This is immediate from [10, Theorem 2.8] since $T \supseteq G_{\mathrm{inn}}$.

(ii) Let $W = \mathbf{C}_G(R)$ so that $W \lhd G$ and $W \subseteq T$. Since char $k = 0$, any nontrivial translation has infinite order as an automorphism and hence T/W is torsion free. Suppose by way of contradiction that $T \cap \Delta^+(G) \neq 1$. Then we must have $W \cap \Delta^+(G) \neq 1$ and since $W \lhd G$ there exists a nontrivial finite normal subgroup N of G with $N \subseteq W$. Set $\alpha = \sum_{x \in N} \bar{x} \in R * N \subseteq R * G$. Since $N \lhd G$,

$N \subseteq W$ and $R * G$ has no twisting, we see that $\alpha \neq 0$ is central in $R * G$. Furthermore $\alpha(\alpha - |N|) = 0$ so α is a zero divisor and $R * G$ is not prime.

Part (ii) above is false in characteristic $p > 0$. A counterexample will be offered at the end of this section.

Let $R * G$ be as above and assume that $T \cap \Delta^+(G) = 1$ so that $G_{\rm inn} \cap \Delta^+(G) = 1$ and the crossed product is prime. Let $0 \neq q \in Q_0(R * G)$. We say that q is a semi-invariant for L and G if there exists a linear functional $\mu : L \to k$ with

$$[l, q] = \mu(l)q \qquad \text{for all } l \in L$$

and a linear character $\lambda : G \to k$ with

$$\bar{x}^{-1} q \bar{x} = \lambda(x) q \qquad \text{for all } x \in G.$$

In other words, q is a common eigenvector for the natural actions of L and of G on $Q_0(R * G)$. We define the semicenter SZ of $R * G$ to be the linear span of all semi-invariants for L and G contained in $R * G$ itself. SZ is easily seen to be a subalgebra of $R * G$.

The following result shows how Theorem A applies to this situation. We continue with the above notation and assumptions. Furthermore we let $S = Q_s(R)$ be the symmetric Martindale ring of quotients of R and $E = \mathbf{C}_{S*G}(S)$.

THEOREM 4.2. *Let $R = U(L)$ be the universal enveloping algebra of the Lie algebra L over k and let $R * G$ be a k-algebra crossed product. Let T be the translation subgroup of G and assume that $T \cap \Delta^+(G) = 1$. If $0 \neq q \in Q_0(R * G)$ is a semi-invariant for L and G then*

(i) *q is a unit of $Q_0(R * G)$ with $q^{-1} R \bar{x} q = R \bar{x}$ for all $x \in G$;*

(ii) *there exists a finitely generated subgroup H of $T \cap \Delta(G)$ with $H \lhd G$ such that q is a unit in $Q_0(R * H_1)$ for all $H \subseteq H_1 \subseteq T \cap \Delta(G)$.*

PROOF. The formula $[l, q] = \mu(l)q$ implies that $lq \subseteq qR$ for all $l \in L$ and thus $Rq \subseteq qR$. Similarly we obtain the reverse inclusion so $Rq = qR$. In addition $\bar{x}^{-1} q \bar{x} = \lambda(x) q$ for all $x \in G$, so we have $R\bar{x}q = qR\bar{x}$. It follows that $0 \neq q \in Q_0(R * G)$ normalizes $R * G$ and hence q is a unit of $Q_0(R * G)$ with $q^{-1} R \bar{x} q = R \bar{x}$ for all $x \in G$. We can now apply Theorem 2.10 with $\sigma = 1$ and $G_0 = G$.

By Theorem 2.10(ii), $q = \alpha^{-1} \beta s \bar{g}$ where $\alpha \in \mathbf{Z}(S * G) \subseteq S * (G_{\rm inn} \cap \Delta)$, $\beta \in E \subseteq S * (G_{\rm inn} \cap \Delta)$, s is a unit of S normalizing R and $g \in G$. Observe that the equation $lq - ql = \mu(l) q$ is equivalent to $q^{-1} l q = l + \mu(l)$ so q acts like a translation on $R = U(L)$. Furthermore, by [9, Theorem 1], s also acts like a translation. Thus since β and α centralize R we see that g acts like a translation.

Therefore by Theorem 2.10(iv)(iii), $g \in T \cap \Delta(G)$ and G normalizes the coset $(G_{\text{inn}} \cap \Delta)g$. Since $G_{\text{inn}} \subseteq T$, it follows from all of this that there exists a finitely generated normal subgroup H of G with $\alpha, \beta, s, \bar{g} \in S * H$ and with $H \subseteq T \cap \Delta$.

Let H_1 be any subgroup of $T \cap \Delta$ containing H. Since $T \cap \Delta^+ = 1$, $T \cap \Delta$ is torsion free abelian and hence $R * H_1$ is prime. Now $\alpha, \beta s \bar{g} \in S * H \subseteq S * H_1$ so there exists $0 \neq I \triangleleft R$ with $I\alpha, I\beta s\bar{g} \subseteq R * H_1$. Thus $(I\alpha)q = I\beta s \bar{g} \subseteq R * H_1$. But q normalizes $R * H_1$ so we have $(R * H_1)(I\alpha)(R * H_1)q \subseteq R * H_1$ and hence we deduce first that $q \in Q_0(R * H_1)$ and then that q is a unit in that quotient ring.

It is convenient to observe a partial converse of the above.

Lemma 4.3. *Let H be a subgroup of $T \cap \Delta(G)$ and let q be a unit of $Q_0(R * H)$ with $q^{-1}R\bar{x}q = R\bar{x}$ for all $x \in H$. Then q is a semi-invariant for L and H.*

Proof. Note that $H \subseteq T \cap \Delta$ implies that H is torsion free abelian and thus $R * H$ is prime. If $x \in H$, then $q^{-1}\bar{x}q = u_x\bar{x}$ with u_x a unit of $R = U(L)$. Thus $u_x \in k \backslash 0$ so $q^{-1}\bar{x}q = \lambda(x)\bar{x}$ for some $\lambda : G \to k$. But then $\bar{x}q\bar{x}^{-1} = \lambda(x)q$ so q is an eigenvector for G. On the other hand, by Lemma 3.1(i), q acts on R in the same way as $s\bar{h}$. Here s is a unit of S normalizing R and $h \in H \subseteq T$, so we know that both s and $\bar{h}$ act as translations. Thus q acts as a translation and the equation $q^{-1}lq = l + \mu(l)$ yields $lq - ql = \mu(l)q$.

We will be able to prove Theorem C from known results on enveloping algebras because of the following observation. It is at this point that we must make some assumption on the nature of the twisting in $R * G$.

Lemma 4.4. *Let $H = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle$ be a finitely generated subgroup of $T \cap \Delta(G)$ and assume that $R * H$ has no twisting. Let X denote the k-linear span of $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ in $R * H$ and let $R[X]$ denote the subring of $R * H$ generated by R and X. Then*

(i) $R[X] = U(X \rtimes L)$, *the enveloping algebra of the Lie algebra $X \rtimes L$, and the elements $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n$ are semi-invariants for $X \times L$;*

(ii) $R * H = R[X]_z$, *the localization of $R[X]$ at the normal element $z = \bar{x}_1\bar{x}_2 \cdots \bar{x}_n$;*

(iii) *if $q \in Q_0(R * H)$ is a semi-invariant for L and H, then $q \in Q_0(U(X \rtimes L))$ and it is a semi-invariant for $X \times L$ centralized by X.*

Proof. (i) Since there is no twisting in $R * H$, the elements $\bar{x}$ with $x \in H$ commute and hence $k[X]$ is the commutative polynomial ring in the n variables $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$. Observe that $H \subseteq T$ so each $\bar{x}_i$ acts like a translation on $U(L)$. Hence $\bar{x}_i^{-1}l\bar{x}_i = l + \mu_i(l)$ so $l\bar{x}_i - \bar{x}_i l = \mu_i(l)\bar{x}_i$. This shows that $X + L$, the direct sum of X and L in $R * H$, is closed under $[\ ,\ ]$ and hence is a Lie algebra. In fact it is $X \rtimes L$, the split extension of X by L. It is now clear from the Poincaré–Birkhoff–Witt theorem that $R[X] = U(X \rtimes L)$.

(ii) Since H is abelian, every element γ of $R * H$ can be multiplied by a suitable power of the element z to clear all negative exponents of group elements. Thus $z^n\gamma \in R[X]$ and $R * H = R[X]_z$. Since conjugation by z clearly normalizes $R[X]$, z is a normal element of this ring.

(iii) Let $q \in Q_0(R * H)$ be a semi-invariant for L and H and set $I = \{\gamma \in R[X] \mid \gamma q \in R[X]\}$ so that $I \neq 0$ by (ii). Now Theorem 4.2 implies that q is a unit of $Q_0(R * H)$ and $q^{-1}R\bar{x}q = R\bar{x}$ for all $x \in H$. Hence q normalizes $R[X]$ and we see that I is in fact a nonzero ideal of $R[X]$. Thus we deduce that q is a unit of $Q_0(R[X]) = Q_0(U(X \rtimes L))$ which gives rise to an X-inner automorphism of the enveloping algebra $U(X \rtimes L)$. Thus q acts like a translation on $X \rtimes L$. As we have seen before, the equation $q^{-1}yq = y + \mu(y)$ for all $y \in X \rtimes L$ implies that $[y,q] = \mu(y)q$ and therefore q is a semi-invariant for $X \rtimes L$. Finally we have $q^{-1}\bar{x}_iq = \bar{x}_i + \mu(\bar{x}_i)$ and $q^{-1}\bar{x}_iq = \lambda(x_i)\bar{x}_i$, since q is a semi-invariant for H, so we conclude that $\mu(\bar{x}_i) = 0$ and hence that X centralizes q.

As a first application we have

THEOREM 4.5. *Let $R = U(L)$ be the universal enveloping algebra of the Lie algebra L over k and let $R * G$ be a k-algebra crossed product. Let T be the translation subgroup of G and assume that $T \cap \Delta^+(G) = 1$ and that $R * (T \cap \Delta)$ has no twisting. Then the semicenter of $R * G$ is a commutative integral domain contained in $R * (T \cap \Delta)$.*

PROOF. Let $0 \neq q \in R * G$ be a semi-invariant for L and G. By Theorem 4.2(ii), $q \in Q_0(R * (T \cap \Delta))$ and thus there exists $0 \neq I \lhd R * (T \cap \Delta)$ with $Iq \subseteq R * (T \cap \Delta)$. Since $R * (T \cap \Delta)$ is prime and $q \in R * G$, a free left $R * (T \cap \Delta)$-module, we see that $q \in R * (T \cap \Delta)$. (This can of course be proved directly and quite easily without all this machinery.) Thus the semicenter is contained in $R * (T \cap \Delta)$ and the latter ring is a domain since R is a domain and $T \cap \Delta$ is torsion free abelian.

Now let q_1, q_2 be two semi-invariants of $R * G$. By the above there exists a finitely generated subgroup H of $T \cap \Delta$ with $q_1, q_2 \in R * H$ and both elements semi-invariants of $R * H$. Since $R * (T \cap \Delta)$ has no twisting, Lemma 4.4 applies and we use its notation. In particular q_1 and q_2 are semi-invariants for $X \rtimes L$ centralized by X and there exist h_1, h_2 in the multiplicative semigroup generated by $x_1, x_2, \ldots, x_n$ with $q_i\bar{h}_i \in U(X \rtimes L)$. By Lemma 4.4(i), each $\bar{x}_j$ is a semi-invariant for $X \rtimes L$ so it follows that each $q_i\bar{h}_i$ is also a semi-invariant. We can now apply Dixmier's theorem [2, Proposition 4.3.5], which is equally valid for infinite dimensional algebras, to deduce that $q_1\bar{h}_1$ and $q_2\bar{h}_2$ commute. Furthermore, by Lemma 4.4(i)(iii), $\bar{h}_i$ commutes with both $\bar{h}_j$ and $q_j\bar{h}_j$ for $i, j = 1,2$. With all of this we conclude immediately that q_1 and q_2 commute.

We remark that some assumption on the twisting is certainly necessary here. For example, let G be a torsion free abelian group and take $R = k$ so that $R * G$ is a twisted group algebra $k^t[G]$. Then every $\bar{x}$ with $x \in G$ is a semi-invariant for $L = 0$ and G. Thus $R * G$ is equal to its semicenter and this ring need not be commutative.

For additional results we must further restrict L to be a finite dimensional Lie algebra over a field k of characteristic zero. The following lemma lists a few known properties of the semicenter of $U(L)$ for such L.

LEMMA 4.6. *Let L be a finite dimensional Lie algebra over a field of characteristic 0 and let SZ be the semicenter of $U(L)$. Then SZ is a unique factorization domain. Furthermore if $0 \neq q \in Q_0(U(L))$ is a semi-invariant for L, then q is contained in the quotient field of SZ. Indeed we can write $q = a/b$ where $a, b \in SZ$ are relatively prime and are both semi-invariants.*

PROOF. We remark that $U(L)$ is an Ore domain so $Q_0(U(L))$ and the field of fractions of SZ both embed in the classical ring of quotients of $U(L)$.

It is shown in [5] and [7] that SZ is a unique factorization domain. Now let $0 \neq q \in Q_0(U(L))$ be a semi-invariant for L and let $0 \neq I \triangleleft U(L)$ with $Iq \subseteq U(L)$. By [4], [6] and [8], I contains a semi-invariant $b_1 \neq 0$. Then $b_1 q = a_1$ is also a semi-invariant contained in $U(L)$, and we see that q is contained in the quotient field of SZ.

Since SZ is a unique factorization domain, we can now write $q = a/b$ where $a, b \in SZ$ are relatively prime and satisfy $a \mid a_1$ and $b \mid b_1$. Furthermore SZ is a domain graded by the torsion free abelian group $\hat{L}$ of linear functionals on L. Thus since a_1 and b_1 are homogeneous elements (that is, semi-invariants for L) and $\hat{L}$ is an ordered group, it follows immediately that a and b must also be homogeneous elements. Thus the lemma is proved.

We can now obtain our main result on crossed products $U(L) * G$.

THEOREM 4.7. *Let $R * G$ be a k-algebra crossed product with $R = U(L)$ the universal enveloping algebra of a finite dimensional Lie algebra L over the field k of characteristic 0. Let T be the translation subgroup of G and assume that $T \cap \Delta^+(G) = 1$ and that $R * (T \cap \Delta)$ has no twisting. Then every semi-invariant for L and G contained in $Q_0(R * G)$ is a quotient of semi-invariants contained in the semicenter SZ.*

Proof. By Theorem 4.5, SZ is a commutative integral domain. Let $0 \neq q \in Q_0(R * G)$ be a semi-invariant for L and G. Then by Theorem 4.2(ii) there exists a finitely generated subgroup H of $T \cap \Delta(G)$ with $H \lhd G$ and with q a unit in $Q_0(R * H)$. Since H is abelian, we can now apply Lemma 4.4 and its notation. In particular $X \rtimes L$ is a finite dimensional Lie algebra and $q \in Q_0(U(X \rtimes L))$. Let A denote the semicenter of $U(X \rtimes L)$ so that A is a unique factorization domain by Lemma 4.6.

Set $W = \mathbf{C}_G(H)$ so that G/W is finite and observe that W normalizes the ring $R[X] = U(X \rtimes L)$. Thus, by [9, Corollary 2], W also normalizes the semicenter A. Since q is a semi-invariant for L and H, it is a semi-invariant for $X \rtimes L$ by Lemma 4.4(iii). Lemma 4.6 now implies that we can write $q = a/b$ where $a, b \in A$ are relatively prime and are both semi-invariants for $X \rtimes L$.

We next observe that a and b are semi-invariants for the group W. To this end let $g \in W$. Then $\bar{g}^{-1}q\bar{g} = \lambda(g)q$ implies that

$$\lambda(g)a/b = (a/b)^{\bar{g}} = a^{\bar{g}}/b^{\bar{g}}.$$

Since conjugation by $\bar{g}$ is an automorphism of A, $a^{\bar{g}}$ and $b^{\bar{g}}$ are also relatively prime so we conclude that $a^{\bar{g}}$ is an associate of a and that $b^{\bar{g}}$ is an associate of b. But the only units of $U(X \rtimes L)$ are in k, so we see immediately that a, b are indeed semi-invariants for W.

Now let $1 = g_1, g_2, \ldots, g_t$ be coset representatives for W in G and set $b_i = b^{\bar{g}_i}$ and $\tilde{b} = b_1 b_2 \cdots b_t$. The goal is to show that $\tilde{b}$ is a semi-invariant for L and G. Note that, since $W \supseteq H$, the above and Theorem 4.2(i) imply that b is a unit of $Q_0(R * H)$ with $b^{-1}R\bar{h}b = R\bar{h}$ for all $h \in H$. Since $H \lhd G$, it therefore follows that for any $g \in G$, $b^{\bar{g}}$ is a unit of $Q_0(R * H)$ with $(b^{\bar{g}})^{-1}R\bar{h}(b^{\bar{g}}) = R\bar{h}$ for all $h \in H$. Hence since $H \subseteq T \cap \Delta(G)$ we conclude from Lemma 4.3 that $b^{\bar{g}}$ is also a semi-invariant for L and H. Thus Lemmas 4.4(iii) and 4.6 imply that $b^{\bar{g}}$ is contained in the field of fractions of A.

We deduce from this observation that $\tilde{b}$ is a unit of $Q_0(R * H)$ and that the b_i's commute. Furthermore, since each b_i is a semi-invariant for L, so is $\tilde{b}$. Now b is a semi-invariant for W so it is easy to see that conjugation by any $\bar{g}$ with $g \in G$ permutes the b_i's up to scalar factors. Thus since the b_i commute, we conclude that $\tilde{b}$ is also a semi-invariant for G. Finally $b \in R * H$ so $\tilde{b} \in R * H$ and we have $\tilde{b}q = \tilde{a} = b_2 b_3 \cdots b_t a \in R * H$. Thus $\tilde{a} = \tilde{b}q$ is also a semi-invariant for L and G contained in SZ and the theorem is proved.

We remark that the semicenter of $R * G$ is rarely a unique factorization domain even when $R = U(L)$ with L finite dimensional over a field of characteristic 0 and $\Delta(G) = 1$. We also note that semi-invariants can certainly be defined in an analogous manner as elements in the maximal ring of quotients $Q_m(R * G)$ or in the classical ring of quotients if it exists. However if q is such a semi-invariant, then we have $q(R * G) = (R * G)q$ and therefore $q \in Q_0(R * G)$ when $R * G$ is prime. Thus we return to the situation already studied. We can now offer the simple

PROOF OF THEOREM C. Let $R = U(L)$ be the universal enveloping algebra of the finite dimensional Lie algebra L over a field k of characteristic 0. We are given a prime k-algebra skew group ring RG. By Proposition 4.1(ii) we then have $T \cap \Delta^+(G) = 1$. With this observation, Theorems 4.5 and 4.7 yield the result.

Finally we translate this material to a result on Hopf algebras. If H is a Hopf algebra and if A is an H-module algebra, then one can define as in [1] the inner action of H on A. A semi-invariant for H is then a common eigenvector for H and the semicenter of H is the linear span of the semi-invariants for the adjoint action of H on itself.

COROLLARY 4.8. *Let H be a prime cocommutative Hopf algebra over an algebraically closed field k of characteristic* 0. *Then the semicenter of H is a commutative integral domain. Furthermore if the primitives of H form a finite dimensional subspace, then every semi-invariant for H in $Q_0(H)$ is a quotient of semi-invariants in the semicenter.*

PROOF. Since k is algebraically closed of characteristic 0, Kostant's theorem ([13, Theorems 1 and 2] and [14, Theorems 8.1.5 and 13.0.1]) implies that $H = U(L)G$, a skew group ring. Here L is a Lie algebra over k and it is the set of primitives of H. By [1], the semi-invariants for H are precisely the semi-invariants for L and G as considered above. Since H is prime, Proposition 4.1(ii) implies that $T \cap \Delta^+(G) = 1$ and then Theorems 4.5 and 4.7 yield the result.

We close with

EXAMPLE 4.9. Let char $k = p > 0$ and let L be the 2-dimensional Lie algebra over k spanned by x and y with $[x, y] = y$. Let σ be the translation automorphism of $U(L)$ given by $x^\sigma = x + 1$, $y^\sigma = y$. Then σ is X-inner of order p. Furthermore if $G = \langle t \rangle$ is cyclic of order p and if t acts on $U(L)$ like σ, then the skew group ring $U(L)G$ is prime even though $G_{\mathrm{inn}} \cap \Delta^+(G) = G \neq 1$.

PROOF. Since $[x, y] = y$ it follows that $y \neq 0$ is a semi-invariant for L in $U(L)$ and hence it is a unit in $Q_0(U(L))$. From $xy - yx = y$ we deduce that

$y^{-1}xy = x + 1$ so σ is the X-inner automorphism determined by y. Since char $k = p > 0$, it is clear that σ has order p.

Now $U(L)G$ is generated over k by x, y, t and hence by x, yt^{-1}, t. But yt^{-1} is central so $U(L)G$ can also be viewed as the skew group ring $k[x, yt^{-1}]G$. But in this case, G is a group of X-outer automorphisms of the commutative polynomial ring $k[x, yt^{-1}]$ so [10, Theorem 2.8] implies that $k[x, yt^{-1}]G$ is prime.

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